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A New Form of Artificial Viscosity*

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An improved form of artificial viscosity results from the substitution of the second Rankine–Hugoniot equation $\{\Delta u = [-\Delta p \Delta(1/\rho)]^{1/2}\}$ in the equation for q. Thus $q = \rho c^2 \mid \Delta u \mid^2$ becomes $q = \rho c^2 \mid \Delta u\, [-\Delta p \Delta(1/\rho)]^{1/2} \mid$. Large velocity gradients (and small pressure gradients) can exist away from shocks, because of geometric effects, and large pressure gradients (and small velocity gradients) can exist in nearly static systems. But if there are both large pressure and velocity gradients, then a shock is present. Thus we see that this form is intrinsically more characteristic of the presence of a shock. Among the several advantages of this form of q, the most important is the improvement in the q heating description as shocks are reflected. Use of the third Rankine–Hugoniot equation in the q is examined, as are some lower-order (linear and 3/2 power) q's.

SYMBOLS AND UNITS

E = specific energy	c = constant (dimensionless)
P = pressure	t = time
U = velocity	q = artificial viscosity
V = volume	η = compression
X = position	ρ = density

Where computer experiment results are quoted, the units (unless otherwise specified) are as follows: $t(\mu\text{sec})$, $U(\text{cm}/\mu\text{sec})$, $X(\text{cm})$, $\rho(\text{g/cm}^3)$, $E(\text{Mbar cm}^3/\text{cm}^3)$, $V(\text{cm}^3)$, q and $P(\text{Mbar})$.

INTRODUCTION

The classic paper by von Neumann and Richtmyer [1] on numerical methods for shocks was published in 1950. Von Neumann and Richtmyer introduced an artificial damping term q to spread the shock so that the hydrodynamic equations

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will not have discontinuities in the solutions for the variables. Thus, for conservation of energy, one obtains (in Lagrangian coordinates)

$$\partial\epsilon/\partial t = -(P + q)(\partial V/\partial t), \tag{1}$$

and for conservation of momentum one obtains

$$\partial U/\partial t = -(1/\rho_0)(\partial/\partial x)(P + q). \tag{2}$$

Both equations are modified by the presence of the damping term q. The continuity equation does not contain the q and is unchanged.

The standard approach is to write an equation for q that implies a viscosity-like property, because physical viscosity will always spread a shock wave. Since [1] was published, it has become apparent that a number of difficulties occur when the standard method is used. One serious difficulty is that the q heating obtained at reflecting boundaries may not be consistent with the Rankine–Hugoniot conditions. Furthermore, the quadratic q for plane waves given by

$$q = \rho(c\,\Delta x)^2(\Delta u/\Delta x)^2 = \rho c^2 \mid \Delta u \mid^2 \tag{3}$$

for zones undergoing compression (otherwise, $q = 0$) does not provide sufficient damping, and the solution is noisy. Frequently a linear q is added to smooth the solution. A linear q lowers the order of the accuracy of the solution, and numerical experiments show that the Rankine–Hugoniot conditions are not satisfied if the linear q contribution becomes large. Furthermore, a linear q frequently introduces oscillations, at material interfaces, which persist after the shock has passed. (Interface noise can also result from the use of "empirical" q forms such as those that use $\partial^2 U/\partial x^2$.)

Cameron [2] has proposed a method for reducing these interface errors, but his technique is not sufficient. To this date, there remains a need for improving the behavior of solutions at boundaries. Thorne and Dahlgren [3] compiled a comparison of different computational techniques, which underlines this need for improving artificial-viscosity calculations.

Theory and Discussion

For the quadratic q term, let us use

$$q = \rho(c\,\Delta x)^2 \mid (\Delta u/\Delta x)[-(\Delta P/\Delta x)(\Delta(1/\rho)/\Delta x)]^{1/2} \mid \tag{4}$$

for zones being compressed; $q = 0$, otherwise. Integrating over a shock, this expression is approximately equivalent to that in [1] because, for a shock, the second Rankine–Hugoniot equation applies:

$$(\Delta u)^2 = -\Delta P\, \Delta(1/\rho). \tag{5}$$

However, in differential form, the q of Eq. (4) is not identical to that in Ref. [1]. (See Appendix A for elaboration of this point.)

It is important to observe that away from shocks the q of Eq. (4) is quite different, when compared to the q of Eq. (3). The q of Eq. (4) is in a form more uniquely associated with shocks because shocks are identified by large velocity changes, large pressure changes, and abrupt density changes. *All* of these effects are present in a shock and, conversely, when all of these effects are observed, a shock is present. It is possible to have any one of the effects alone and not have a shock. For example, large velocity gradients can exist because of geometrical effects. Equation (4) would intrinsically treat an adiabatically squeezed sphere correctly. No significant q would develop, because the pressure gradient would be small, even though a large velocity gradient might be present. Codes frequently avoid q's in these situations by testing

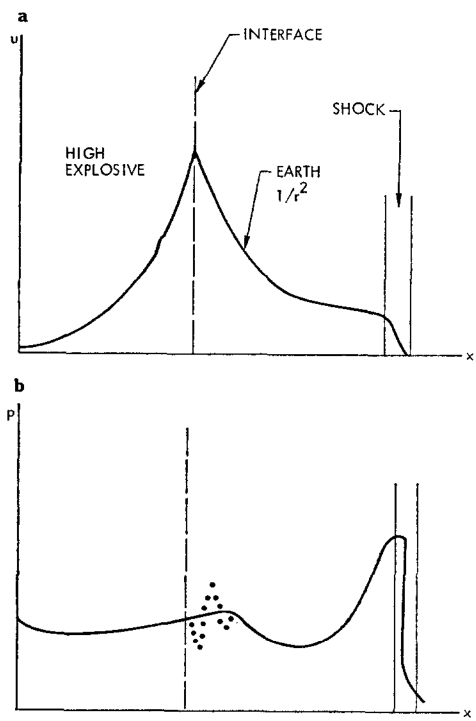

FIG. 1. (a) Velocity vs position. (b) Pressure vs position. The dotted line represents the noise that can be introduced by a linear term.

on velocity gradients or other quantities. However, it seems preferable to have a q whose very form avoids these problems. Another example (as shown in Figs. 1a and 1b) would be the description of a spherical charge of high explosive detonated in the earth. After the shock has passed the earth/high-explosive interface, a steep velocity gradient develops (proportional to $1/r^2$ in the earth), but the pressure gradient is small.

We are now using in Eq. (4) a mix of gradients of zone quantities (pressure and density^{-1}) and grid quantities (velocity). When a perfectly reflecting boundary is reached, a mirror zone is required. Boundary conditions are imposed on both zonal and grid quantities in the calculation of q. Example calculations are presented here that show that the Rankine–Hugoniot conditions are considerably improved at reflecting walls.

Within the shock, perturbations will be damped out proportional to $e^{-\alpha t}$ (in which α is a real number) for the quadratic q. However, the value of α may be quite small near the edges of the shock wave, and additional damping may be required. This result can be obtained by using a lower-order q such as a linear one. Let us try a 3/2 power q given by

$$q_{3/2} = \rho c^{3/2} W^{1/2} \mid \Delta u[-\Delta P \, \Delta(1/\rho)]^{1/2} \mid^{3/4}, \tag{6}$$

where W is a function with dimensions of velocity. (Appendix B examines the stability of a 3/2 power q.) The stability analysis of q requires that the determinant resulting from a particular set of equations equals zero. This determinantal equation should change continuously in going from the shocked region to the unshocked region. This is equivalent to requiring that the damping term α should be continuous in going from the shocked to the unschocked region. The quadratic q and 3/2 power q (see Appendix B) both satisfy this condition; but the linear q does not, unless it is used in the shocked and unshocked regions. This, however, has the undesirable consequence of introducing more artificial dissipation.

Although a linear q has a low-order accuracy and its stability analysis leads to a discontinuity in the determinantal equation, there are other arguments that make it attractive. For an elastic solid, the ratio q/P goes to zero for a quadratic q as the pressure approaches zero, whereas it approaches a constant multiplied by $\Delta P/P$ for a linear q (see Appendix C). UNEC [4], the first code applied to nuclear explosion interactions with earth materials, used a linear q in addition to the quadratic q [5] to avoid this problem. However, a small multiplier is desirable for the best accuracy.

It will be noted that, in the limit of infinitely weak shocks (sound waves), the relation

$$\Delta u = [-\Delta P \, \Delta(1/\rho)]^{1/2} \tag{7}$$

becomes

$$\Delta u = (\Delta P/\rho)(\Delta\rho/\Delta P)^{1/2} = \Delta P/\rho SS, \tag{8}$$

where SS is the speed of sound, which is just the solution of the characteristic equation for sound waves. The author has tried this relation in q equations, and it works quite well for weak shocks. Weak shocks in this case had pressure changes of more than 10 times the preshock pressure. The form in Eq. (8) is simpler to program and could be of value in a two-dimensional or Eulerian grid (an application that comes to mind is the simulation of air-frame dynamics). This will be the subject of a brief report in the future and will consider strong shocks as well.

The use of the second Rankine–Hugoniot condition naturally suggests the use of the third Rankine–Hugoniot condition:

$$\Delta E = -\bar{P}\,\Delta(1/\rho), \tag{9}$$

where $\bar{P}$ is the average of the pressure behind the shock and the pressure in front of the shock. Thus one might try using

$$\Delta U = [(\Delta P/P)\,\Delta E]^{1/2} \tag{10}$$

in the formulation for q or some combination of Eqs. (7) and (10).

Programming

For zones at material interfaces, the term $[-\Delta P\,\Delta(1/\rho)]^{1/2}$ is obtained by a one-sided extrapolation.

For a reflecting boundary, an imaginary zone must be added, and its physical state must be identical to that of the boundary zone. These modifications in the coding have not increased the running time of test problems by as much as 1 %. Problems have been run using $[-\Delta P\,\Delta(1/\rho)]^{1/2}$ to replace Δu completely in the q equation, and they are stable but quite noisy. All problems were run on UKO, a modified version of the elastic-plastic KO code [6].

KO currently uses $q_L = \rho c\,\Delta x(\Delta u/\Delta x)(P/\rho)^{1/2}$ for a linear term; that is, it uses $(P/\rho)^{1/2}$ to get a velocity term rather than using a constant or the sound speed. UKO has not followed this example for the linear term but instead uses the sound speed for W. This insures the damping of shocks in an elastic medium.

A number of "time-selection" schemes have been tried to investigate the importance of achieving centering for the viscosity in the energy and momentum equations. The velocity, pressure, and density were used from the previous cycle (and half cycle) and also from two cycles previous in a variety of combinations. None of these schemes evidenced any significant differences. The scheme selected was that which was the simplest, i.e., use whatever values of velocity, pressure, and density that exist in the code at the time the viscosity is calculated. There was no attempt to iterate on the pressure at the current time step.

Let us consider one approach that might be applicable for an Eulerian formulation. Assume that material A is the material taking up the largest fraction of a zone volume, but that there is none in a neighboring zone. One might use the pressure in the neighboring zone to calculate what the density of A would be if it were present. Then this imaginary density would be used to calculate $\Delta(1/\rho)$. Of course, if neighboring zones contain a material common to both, $\Delta(1/\rho)$ can be obtained directly.

Problem Examples

A number of problem results from UKO calculations are presented here. The first group of examples were run with two different q's:

$$q_u = \rho c_1{}^2 \mid \Delta u \mid^2 + \rho c_2^{3/2}(P/\rho)^{1/4} \mid \Delta u \mid^{3/2}, \tag{11}$$

and

$$q_p = \rho c_1{}^2 \mid \Delta u \mid \mid[-\Delta P\, \Delta(1/\rho)]^{1/2} \mid + \rho c_2^{3/2}(P/\rho)^{1/4} [\mid \Delta u \mid \mid[-\Delta P\, \Delta(1/\rho)]^{1/2} \mid]^{3/4}. \tag{12}$$

The same coefficients are used to spread the shock over the same number of zones in each case, thereby allowing direct comparisons. For convenience and brevity, the first will be called the velocity q and the second the pressure q. Otherwise, comparisons were made between identical problems. It is worth noting that q_u and q_p responded experimentally to the same stability condition.

The first examples will be for the case of a plane wave shock. Consider a 30-cm length of ideal gas with $\gamma = 1.4$ and one end against an immovable wall. At the other end apply a constant pressure of 1.0 Mbar. Let the gas have an initial density of 0.008 g/cm³ and a pressure of 0.00004 Mbar. This is a good approximation to an infinitely strong shock. Let $c_1{}^2 = 4$ and $c_2^{3/2} = 0.4$ with zone size $\Delta x = 0.5$ cm (60 zones). After 2.72 μsec, the shock has bounced off the wall. Table I shows the pressure profile of the shock as calculated by the two different q's.

We know analytically that, for an infinitely strong shock in an ideal gas,

$$P_r/P_i = (3\gamma - 1)/(\gamma - 1)(=8 \text{ for } \gamma = 1.4), \tag{13}$$

where P_r is the reflected shock pressure and P_i is the incident shock pressure. We note that the q_p solution is both smoother and more accurate. The shock seems to be slightly sharper, although this effect is not particularly significant.

TABLE I

J (zone)	P (Mbar) for q_u	P (Mbar) for q_p
40	7.91	7.99
41	7.99	7.99
42	7.91	7.98
43	7.96	7.99
44	7.96	7.98
45	7.45	7.83
46	5.13	6.57
47	3.03	4.44
48	1.74	2.49
49	1.20	1.33
50	1.03	1.03
51	0.99	1.00
52	1.00	1.00

The zone values near the wall are given by Table II. Zone $J = 1$ is next to the reflecting wall.

In Table II we observe the anomalous heating introduced by the q, but we also observe that the effect is much worse in the q_u case. For the q_p problem, the largest error for E (E should $= 0.95$) is about 15 %, but in the q_u problem the largest error is about 60 %. Thus we see that the Rankine–Hugoniot conditions are preserved better by the use of the pressure q. The q_p calculation

TABLE II

J	For q_u			For q_p		
	P (Mbar)	$E\left(\frac{\text{Mbar-cm}^3}{\text{cm}^3}\right)$	η	P (Mbar)	$E\left(\frac{\text{Mbar-cm}^3}{\text{cm}^3}\right)$	η
1	7.94	1.55	12.8	7.99	0.91	21.8
2	7.96	1.16	17.2	7.99	1.10	18.2
3	7.96	0.88	22.4	7.99	0.92	21.7
4	7.96	0.79	25.1	7.99	0.95	21.1
5	7.95	0.79	25.1	7.99	0.93	21.6
6	7.92	0.80	24.7	7.98	0.94	21.1
7	7.96	0.82	24.3	7.99	0.94	21.1
8	7.94	0.83	23.8	7.99	0.95	21.0
9	7.91	0.85	23.2	7.99	0.95	21.0
10	7.96	0.88	22.6	7.99	0.95	21.0

comes to equilibrium much quicker than the q_u does; that is, the E and η values stabilize very near the reflecting wall. The values for E and η for the q_u solution stabilize and approach those of the q_p solution but at a much greater distance from the wall (larger J value). Again we note that the q_p solution is both smoother and more accurate. We ran a weak-shock calculation, and the results were similar.

For our next example, let us investigate a spherically converging shock of infinite strength in an ideal gas of $\gamma = 1.4$. Use the same initial fluid conditions as in the previous problem, with a 1-Mbar pressure at the outside. Let the outer radius be 30 cm with a constant Δx ($= 0.3$ cm) from the outside into a 6-cm radius. Let $\Delta x = 0.01$ cm at the origin, and let the size of zones increase by a constant percentage ($\sim$10 %) until the zone at 6 cm radius is 0.3 cm thick. We will examine the problem solution near the origin, where the approximation of an infinitely strong shock is valid. Let $c_1^2 = 4$ and $c_2^{3/2} = 0.4$ again.

The analytic solution is due to Guderley [7]. For $\gamma = 1.4$ we expect $P_r/P_i = 26$ [8] for a fixed point in space; we also expect that Pt^2/x^2 [9] will be a constant, where P is the pressure of the shock when the shock is at position x at time t. The time is negative for convergence, zero when the shock is at the center, and positive for divergence. The pressure ratio (reflected to incident values) is not a particularly sensitive function; both q_u and q_p give approximately the same answers ($P_r/P_i \simeq 23$), even though the answers to the two problems are noticeably different in the amount of heating done to the zones in the center.

Before the results of problems with the different q's are compared, it should be pointed out that care must be taken evaluating the pressure following a converging

TABLE III

	For q_u			For q_p		
J	P (Mbar)	$E\left(\frac{\text{Mbar-cm}^3}{\text{cm}^3}\right)$	η	P (Mbar)	$E\left(\frac{\text{Mbar-cm}^3}{\text{cm}^3}\right)$	η
1	1048	242	10.8	891	127	17.5
2	1058	175	15.1	893	143	15.5
3	1077	120	22.5	896	116	19.4
4	1104	92	30.0	903	98	23.0
5	1133	85	33.5	914	85	26.9
6	1160	77	37.8	926	77	30.0
7	1170	69	42.4	932	70	33.5
8	1158	64	45.1	930	63	37.0
9	1130	59	47.6	921	57	40.0
10	1110	54	51.2	911	53	47.2

shock in spherical coordinates, because there is an adiabatic pressure increase following the shock (due to convergence).

In Table III we can examine zonal values near the center of the sphere after the shock has been reflected (at a time when the reflected shock is at a radius of 0.17 cm). The two solutions are quite different, and we might surmise that the anomalous q heating is much improved by the pressure q. The flatter pressure distribution of the q_p solution also appears more believable, but such observations are not all conclusive.

However, we know that Pt^2/r^2 must be a constant for the shock, and in Fig. 2 this quantity is plotted for both the converging and diverging shocks. Note that the scales for the Pt^2/r^2 axes are different for the converging shock and the diverging shock. The solution is clearly much more favorable for the q_p case.

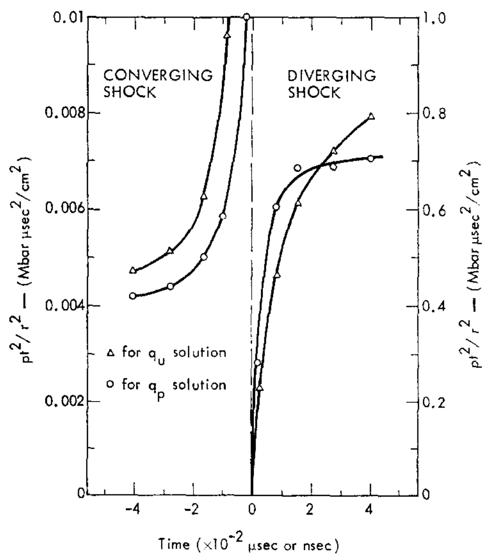

FIG. 2. (Pt^2/r^2) vs t.

Some spherical problems were also run with weak shocks generated at the outer surface, and they displayed qualitatively similar properties.

For a final example, consider a 10-cm-radius sphere of high explosive at the center of a 100-cm-radius sphere of iron; let the explosive be ignited throughout its volume at the start of the problem, with $\Delta x = 1$ cm. Table IV presents the pressure profile of the shock when it has nearly reached the outer boundary of the iron sphere. Also shown in Table IV is the region of the explosive-iron interface at the same time (time $= 178\,\mu$sec). Here $c_1{}^2 = 5.5$ and $c_2^{3/2} = 3.0$ (more damping is usually

TABLE IV

J	X (cm)	P (kbar) for q_u	P (kbar) for q_p	
94	94.0	3.95	3.49	
93	93.0	6.13	5.72	
92	92.0	8.32	8.15	
91	91.0	10.1	10.2	
90	90.0	11.2	11.5	
89	89.0	11.6	11.9	
88	88.0	11.5	11.7	
87	87.1	11.0	11.1	
86	86.1	10.5	10.4	
85	85.1	10.0	9.9	
—		—	—	Iron
—	—	—	—	
—	—	—	—	
15	19.0	5.6	5.8	
14	18.4	5.4	5.7	
13	17.8	5.5	5.6	
12	17.3	5.1	5.5	
11	16.5	4.8	5.3	
		Material interface		
10	14.8	5.1	5.1	
9	13.1	5.0	5.1	
8	11.4	5.1	5.1	High explosive
7	9.7	5.1	5.1	
6	8.0	5.1	5.1	

required for spherically diverging waves in solids). The q_p solution is better, in that the shock is sharper and the pressure profile is smoother at the interface.

A wide variety of other problems (and materials) were examined, all of which showed improvement through the use of q_p. A plane-wave problem was run with a pressure profile on each end; and, after the shocks collided, the result was identical to the result of the rigid-wall problem. Planar problems with many materials (gases and solids) were run to examine the anomalous heating at interfaces; in every case, the q_p result was a considerable improvement over the q_u result. It should be noted that all the above problems avoided interfaces with large differences in zone size. Problems were run with large zone-size changes at interfaces, and the anomalous heating was a serious problem for both forms of the q (although slightly less troublesome for q_p).

The previous problem examples were run basically with a quadratic term, while the 3/2 power term could be considered as a perturbation term (admittedly large in the last example) to obtain smoothing. This meant that all the analysis in Ref. [1] applied reasonably well. We have also seen that the 3/2 power term has desirable damping properties by itself.

We will now examine some problems that have only one q term, in order to isolate specific effects and to demonstrate that the 3/2 power q alone is sufficient for many problems. The problem example used here is the same as the first: a plane strong shock in gas with $\gamma = 1.4$. Table V gives the results of using several different q's. The constants were chosen to provide shock widths of the same size. The results can also be compared to those of Table II.

TABLE V

J	$q = 4\rho \mid \Delta u \mid^2$		$q = 3\rho(P/\rho)^{1/4} \mid \Delta u \mid^{3/2}$		$q = 3\rho(P/\rho)^{1/4} \mid \Delta u \mid^{3/4} \mid [-\Delta P \Delta(1/\rho)]^{1/2} \mid^{3/4}$	
(wall)	P	E	P	E	P	E
1	7.959	1.54	7.974	1.59	7.994	0.96
2	7.953	1.13	7.974	1.07	7.993	1.13
3	7.954	0.86	7.974	0.96	7.993	0.95
4	7.938	0.78	7.975	0.83	7.992	0.96
5	7.938	0.79	7.974	0.84	7.992	0.94
6	7.935	0.80	7.974	0.86	7.992	0.95
7	7.925	0.82	7.975	0.87	7.991	0.95
8	7.930	0.84	7.974	0.89	7.992	0.95
9	7.937	0.86	7.974	0.90	7.992	0.95
10	7.932	0.87	7.974	0.92	7.993	0.95

First we note that the 3/2 power term in $\mid \Delta u \mid$ has provided considerable smoothness and slightly more accuracy than the quadratic term provides. This results from the fact that the 3/2 power term provides more damping near the trailing edge of the shock. However, going to 3/2 power in Δu has not helped the heating at the wall.

The heating at the wall has been improved by the Δu mixed with $[-\Delta P\, \Delta(1/\rho)]^{1/2}$ (and the accuracy in the pressure column has also been slightly improved). One might anticipate that the 3-point difference scheme for ΔP or $\Delta(1/\rho)$ is a reason for the big improvement and replace Δu by $[-\Delta P\, \Delta(1/\rho)]^{1/2}$ entirely. However, the results of such a calculation are similar to the pure $\mid \Delta u \mid$ examples in Table V, although the calculation is noisier. This supports the idea that it is the mix of zone and grid quantities that provides the improvement.

Plane-wave ideal-gas ($\gamma = 1.4$) problems have been run with a small q coefficient, specifically using

$$q = 2\rho(P/\rho)^{1/4} \mid \Delta u \mid^{3/4} \mid [-\Delta P\, \Delta(1/\rho)]^{1/2} \mid^{3/4}, \tag{14}$$

and the results are extremely smooth and accurate (almost as good as the right-hand column of Table V). However, the pressure values at the shock give a much sharper shock (by a factor of approximately 2/3). The accuracy of the values near the shock is still excellent and virtually noise-free. Cutting the multiplier in half (from 2 to 1) does, however, lead to a noisy and less satisfactory solution. A smaller multiplier means savings in machine time.

This paper has not examined the linear q. However, it should be mentioned that problems have been run using it, and their performance is also improved by mixing zone and grid quantities via the second Rankine–Hugoniot condition.

Use of the Third Rankine–Hugoniot Equation

Up to this point, we have seen that shock shapes are relatively unaffected by use of the second Rankine–Hugoniot equation. This might have been anticipated, since Δu and $[-\Delta P\, \Delta(1/\rho)]^{1/2}$ could be expected to show similar functional behavior in a shock. However, the third Rankine–Hugoniot equation is

$$\Delta E = -((P_1 + P_2)/2)\, \Delta(1/\rho), \tag{15}$$

where P_1 and P_2 are the pressure values of the regions before and after the shock. These values (P_1 and P_2) are really unknowns, and the replacement of $1/2\,(P_1 + P_2)$ by some quantity such as P will introduce an uncertain functional behavior into a q. In fact, it will be seen that the shape of the shock can be significantly affected.

A wide variety of problem types have been run, and they indicate that the third Rankine–Hugoniot equation provides improvement in the anomalous heating at boundaries and interfaces. Again, the best choice for q is indicated experimentally to be an equal mix of zone quantities and grid quantities.

Then a 3/2 power q would have the following form:

$$q = \rho c^{3/2} W^{1/4} \mid \Delta u \mid^{3/4} \mid [\Delta P(-\Delta(1/\rho)\, \Delta E/P)^{1/2}]^{1/2} \mid^{3/4}. \tag{16}$$

The results of this q in a plane strong shock for $\gamma = 1.4$ (like the results of Table V) are given in Table VI and are compared to the best q of Table V.

The accuracy is considerably improved at the wall; the size of the worst error in the calculation of E has been reduced by a factor of $2\frac{1}{2}$. At the same time, we note that the shock is sharper and of a considerably different shape for the reflected shock shown in Table VI. However, the incident shock did not have a significantly different shape. Weak shock problems also did not display significantly different shock shapes for the different q forms.

Approximating $\frac{1}{2}(P_1 + P_2)$ by some quantity such as P might be found to have adverse effects for some material equations of state, although this has not yet been

TABLE VI

	$q=3\rho(P/\rho)^{1/4}\lvert\Delta u\rvert^{3/4}\lvert[-\Delta P\Delta(1/\rho)]^{1/2}\rvert$		$q=3\rho(P/\rho)^{1/4}\lvert\Delta u\rvert^{3/4}\lvert[\Delta P-\Delta(1/\rho)(\Delta E/P)^{1/2}]^{1/2}\rvert^{3/4}$	
J	P	E	P	E
51	1.000	0.417	1.000	0.417
50	1.005	0.420	1.004	0.417
49	1.065	0.427	1.058	0.424
48	1.456	0.484	1.565	0.497
47	2.802	0.652	4.040	0.756
Shock 46	4.728	0.804	6.555	0.900
45	6.506	0.896	7.750	0.947
44	7.571	0.938	7.981	0.955
43	7.945	0.951	7.957	0.954
42	7.992	0.952	7.985	0.954
41	7.994	0.952	7.976	0.953
—	—	—	—	—
—	—	—	—	—
—	—	—	—	—
10	7.993	0.951	7.998	0.950
9	7.992	0.950	7.997	0.949
8	7.992	0.950	7.997	0.947
7	7.991	0.945	7.996	0.944
6	7.992	0.946	7.996	0.939
5	7.992	0.940	7.997	0.938
4	7.992	0.964	7.997	0.937
3	7.993	0.955	7.998	0.948
2	7.993	1.140	7.998	1.03
Wall 1	7.994	0.963	7.998	0.929

observed. Equation (16) also provides less damping at the trailing edge of the shock and results in a noisier solution, which can be smoothed by adding more damping (such as a lower-order of q).

Conclusions

Empirically derived forms of artificial viscosity have been described that are uniquely associated with shocks; and, for the problems investigated, they provide smoother and more accurate answers, in comparison with methods given in Ref. [1]. They improve solution behavior at interfaces, and, in particular, they improve anomalous heating by a considerable factor at walls. Whether a quadratic term, 3/2 power term, linear term, some other term, or a combination of terms should be used in a calculation depends on the types of materials being modeled and the pressure regime of interest. Some conclusions I've reached regarding the use of these artificial viscosity forms are presented in Appendix D.

A weak-shock version could be used easily in some Eulerian codes, but clever programming would be required for an Eulerian code capable of describing strong-shock systems; still, one suspects it can be done, just as one might also expect benefits in multidimensional Lagrangian codes. The use of Rankine–Hugoniot relations in the difference equations for shock-sensitive forms may be widely useful, for example, in conservative schemes.

Appendix A

The q Eqs. (3) and (4) would be identical if

$$\left(\frac{\partial U}{\partial x}\right)^2 = -\frac{\partial P}{\partial x}\frac{\partial(1/\rho)}{\partial x}. \tag{A1}$$

Consider a coordinate system moving with the shock; then ω is the distance from the shock where

$$\omega = x - st, \tag{A2}$$

where s is the shock speed. Then we would have identical forms if

$$(dU/d\omega)^2 = -(dP/d\omega)(dV/d\omega), \tag{A3}$$

where $V = 1/\rho$. But we know from the continuity equation that

$$-M(dV/d\omega) = (dU/d\omega), \tag{A4}$$

where $M = \rho_0 s$ (ρ_0 is the density in undisturbed regions). Thus the forms are the same if

$$-M^2(dV/d\omega) = (dp/d\omega). \tag{A5}$$

In contradiction, however, the equation of motion [Eqs. (2)] and (A4) gives the result

$$-M^2(dV/d\omega) = (d/d\omega)(p + q). \tag{A6}$$

For the solution by von Neumann and Richtmyer [1], $\int (dq/d\omega)\, d\omega = 0$ when evaluated over the shock. This will be true for any q that equals zero at the shock edges. Thus one is not surprised that the new q gives results like those of Ref. 1. In fact, numerical experiments with the pressure q produce shock-wave shapes (pressure vs ω, density vs ω, etc.) that are virtually indistinguishable from the traditional q given by von Neumann and Richtmyer.

Appendix B

Consider perturbations δU, δV, δP, and δq on the desired solutions for a γ-law gas. Assume these perturbations are of the form

$$\delta U = \delta U_0 e^{ikx+\alpha t}, \qquad \delta V = \delta V_0 e^{ikx+\alpha t}, \text{ etc.,} \tag{B1}$$

where δU_0, δV_0, δP_0, δq_0, k, and α are constant and k is real. We then obtain from the equations of continuity, momentum, and energy and the equation for q the following set of linear equations:

$$\alpha \rho_0 \, \delta V_0 - ik \, \delta U_0 = 0, \tag{B2}$$

$$ik \, \delta q_0 + \alpha \rho_0 \, \delta U_0 + ik \, \delta P_0 = 0, \tag{B3}$$

$$(\gamma - 1) \frac{\partial V}{\partial t} \delta q_0 + \left[\frac{\partial P}{\partial t} + \alpha\gamma P + \alpha(\gamma - 1) \frac{(c\,\Delta x)^{3/2}}{V} \left|\frac{\partial U}{\partial x}\right|^{1/2} \frac{\partial U}{\partial x}\right] \delta V_0$$
$$+ \delta P_0 \left(V\alpha + \gamma \frac{\partial V}{\partial t}\right) = 0, \tag{B4}$$

$$-\delta q_0 + W^{1/2} \frac{(c\,\Delta x)^{3/2}}{V^2} \frac{\partial U}{\partial x} \left|\frac{\partial U}{\partial x}\right|^{1/2} \delta V_0 - \frac{3}{2} ik \frac{(c\,\Delta x)^{3/2}}{V} W^{1/2} \left|\frac{\partial U}{\partial x}\right|^{1/2} \delta U_0 = 0, \tag{B5}$$

where we have used

$$q = -[(c\,\Delta x)^{3/2}/V] \, W^{1/2} \, |\partial U/\partial x|^{1/2} (\partial U/\partial x) \tag{B6}$$

for a shock traveling in the positive x direction. (Thus q is positive, since $\partial U/\partial x$ is negative.) The determinant of the above must equal zero in order to have a solution. We then obtain the determinantal equation:

$$0 = -(\alpha\rho_0)^2 \left(V\alpha + \gamma \frac{\partial V}{\partial t}\right) + (\gamma - 1)\, \alpha\rho_0 \frac{3}{2} \frac{(c\,\Delta x)^{3/2}}{V} \left|\frac{\partial U}{\partial x}\right|^{1/2} W^{1/2} k^2 \frac{\partial V}{\partial t}$$
$$- \alpha\rho_0 k^2 \frac{3}{2} \frac{(c\,\Delta x)^{3/2}}{V} \left|\frac{\partial U}{\partial x}\right|^{1/2} - k^2(\gamma - 1) \frac{\partial V}{\partial t} \frac{(c\,\Delta x)^{3/2}}{V^2} W^{1/2} \frac{\partial U}{\partial x} \left|\frac{\partial U}{\partial x}\right|^{1/2}$$
$$+ k^2 \left(V\alpha + \gamma \frac{\partial V}{\partial t}\right) \frac{(c\,\Delta x)^{3/2}}{V^2} \frac{\partial U}{\partial x} \left|\frac{\partial U}{\partial x}\right|^{1/2}$$
$$- k^2 \left[\frac{\partial P}{\partial t} + \alpha P + \alpha W^{1/2} (\gamma - 1) \frac{(c\,\Delta x)^{3/2}}{V} \left|\frac{\partial U}{\partial x}\right|^{1/2} \frac{\partial U}{\partial x}\right]. \tag{B7}$$

If we restrict ourselves to the dominant terms (highest order in k and α), we get

$$\alpha = -(3/2)\, k^2 [(c\,\Delta x)^{3/2}/\rho_0 V] \, W^{1/2} \, |\partial U/\partial x|^{1/2} \quad \text{in shocked regions} \tag{B8}$$

and

$$\alpha^2 = -(k^2\gamma P/\rho_0{}^2 V) \quad \text{in normal regions.} \tag{B9}$$

Thus perturbations are damped out in the shock, but they are propagated without change in normal regions. This result is like von Neumann and Richtmeyer's [1] for the quadratic q. A linear q would have obtained a value for α in the shock region proportional to $|\partial U/\partial x|^0$, which does not go to zero at the shock boundaries. (At this point, it is appropriate to restate that the purpose of the q is to eliminate the discontinuities in the solutions of the differential equations, so that difference equations can be applied.) It would seem that the decay constant α is one solution variable (even though a numerical artifact) that should be continuous. Thus we would prefer to avoid a linear q if at all possible. The same kind of analysis can be performed for

$$q = \frac{(c\,\Delta x)^{3/2}}{V} W^{1/2} \left[\frac{\partial U}{\partial x}\left(-\frac{\partial P}{\partial x}\frac{\partial(1/\rho)}{\partial x}\right)^{1/2}\right]^{3/2},$$

with similar results.

Appendix C

An elastic solid is characterized by an equation of state (E.O.S.) for which the pressure is a function of only the volume. The following equation defining the bulk modulus (K) is a standard way to express the E.O.S.:

$$K = -V(dP/dV), \tag{C1}$$

where V is the specific volume ($1/\rho$) and P is the pressure. The E.O.S. is complete if K is given as a function of volume.

Then, for the quadratic q we obtain

$$q/P = \rho c^2 \,|\, \Delta P\, \Delta V \,|/P. \tag{C2}$$

But $\Delta V = -V\,\Delta P/K$ from Eq. (C1), hence

$$q/P = (V\rho c^2/K)(\Delta P/P)\,\Delta P, \tag{C3}$$

which approaches zero as ΔP approaches zero.

For the linear q we have

$$q_L/P = (SS)\,\rho c\,\Delta u/P, \tag{C4}$$

where SS is the sound speed. Then, using Eq. (C1) we get

$$q/P = (V^{1/2} SS \rho c / K^{1/2})(\Delta P/P). \tag{C5}$$

This function is well behaved, in the sense that it is only the relative value of ΔP to P that matters.

APPENDIX D

Viecelli [10] has recently shown how a linear q determines the form of the solution for spherically decaying shocks in a solid. This work gives ample evidence of the need to minimize the linear q coefficient. In Viecelli's analysis, the linear q is calculated for positive pressures, regardless of the sign of $\partial u/\partial x$. (Thus, q_L is negative for expanding zones.) I have found it convenient to implement the linear q in the same fashion; but q is set equal to zero if its absolute value is less than a given fraction (typically set $=0.001$) of the pressure. For most problems, $c^2 = 2$ (for the quadratic multiplier) and $c_L = 0.2$ proves quite adequate. This provides sharp shocks (a smaller c^2 than was used in the earlier examples) that are still smooth, and yet the c_L is not too large for most calculations. Then we have

$$\left.\begin{array}{lll}
\text{I.} & \text{for} & P \leqslant 0.0, \\
 & & q = 0.0 \\
\text{II.} & \text{for} & P > 0.0, \\
 & \text{for} \quad dV < 0 & \left\{\begin{array}{l} q = 2.0\rho\, \Delta u(-\Delta P\, \Delta 1/\rho)^{1/2} \\ \qquad + 0.2(SS)\rho\, |\Delta u|^{1/2}\, (-\Delta P\, \Delta 1/\rho)^{1/4} \end{array}\right. \\
 & \text{for} \quad dV \geqslant 0 & \{q = -0.2(SS)\rho\, |\Delta u|^{1/2}\, (-\Delta P\, \Delta 1/\rho)^{1/4} \\
\multicolumn{3}{l}{\text{If } |q| < \text{fraction (typically } 0.001)P,} \\
 & & q = 0.0.
\end{array}\right\} \tag{D1}$$

Furthermore, I have observed that using a large Δt improves the anomalous heating slightly, provided that the stability condition is not violated. The form recommended here is quite similar to the one published in [6].

Let

$$\Delta t = 2/3\, \{\Delta x/[(SS)^2 + b^2]^{1/2}\}, \tag{D2}$$

where $b = 2c^2\, \Delta x(\dot{V}/V)$, and SS is the sound speed.

Using this Δt control and the q multipliers given in Eq. (D1), the largest error in the anomalous heating effect for the problem described in Tables I, II, V, and VI was only 8 % (similar to the best result in Table VI).

These recommendations are obviously subjective and reflect my set of calculations and my bias regarding shock sharpness, smoothness, accuracy, computer time, etc.

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